Unit → 3

(1) System of linear Eq^n

(2) Numerical differentiation & Numerical integration

(3) Sol^n of differential Eq^n

System of Linear Eq^n →

(a) **Linear Eq^n →** A linear Eq^n involving two variables has the std. form.

$$ax + by = c$$

where $a, b, c \in R$

Ex → $x + 4y = 0$ (L.E in 2 variables)

$x = 0$ (L.E in 1 variable)

$x + y + z = 12$ (L.E in 3 variables)

(b) **Result →** $x + y + z = 10$

$x + y + 2z = 20$

System of Eq^n →

Given -

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Write the given system of Eq^n into the matrix form

$$AX = B$$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$\frac{A}{B}$ matrix →

$$\frac{A}{B} = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

Gauss elimination method →

Working Procedure → Consider the following system of Eq^n

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3.$$

Step 1 → $(a_1 \neq 0)$, If $a_1 = 0$ then interchange the Eq^n.

Step 2 → write the given system of Eq^n into the $AX = B$ form

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Now, we find (A/B) form

x & y

$$(A/B) = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right] \begin{array}{l} — R_1 — (1) \\ — R_2 — (2) \\ — R_3 — (3) \end{array}$$

Step 3 → Eliminate x from Eq^n (2) & (3) by using Eq^n (1)

$$R_2 \leftarrow R_2 \pm R_1$$
$$R_3 \leftarrow R_3 \pm R_1$$

$$[A/B] = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2 & c_2 & d_2 \\ 0 & b_3 & c_3 & d_3 \end{array}\right]$$

Step 4 → Eliminate y from Eq^n (3) by using Eq^n (2)

$$R_3 \leftarrow R_3 \pm R_2$$

$$[A/B] = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2 & c_2 & d_2 \\ 0 & 0 & c_3 & d_3 \end{array}\right]$$

Step 5 → calculate the value of unknown variables by using backward substitution.

Q → Solve the following system of Eq^n by using Gauss Elimination method -

$$2x + 3y - z = 5 \quad — (1)$$
$$4x + 4y - 3z = 3 \quad — (2)$$
$$2x - 3y + 2z = 2 \quad — (2)$$

Step 1 → first we write the given system of Eq^n into the $AX = B$ form.

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$$

Now, we find $[A/B]$ form

$$[A/B] = \begin{bmatrix} 2 & 3 & -1 & | & 5 \\ 4 & 4 & -3 & | & 3 \\ 2 & -3 & 2 & | & 2 \end{bmatrix}$$

Step 2 → Eliminate x from Eq^n ② & ③ by using ①

$$R_2 \leftarrow R_2 - 2R_1$$
$$R_3 \leftarrow R_3 - R_1$$

$$[A/B] = \begin{bmatrix} 2 & 3 & -1 & | & 5 \\ 0 & -2 & -1 & | & -7 \\ 0 & -6 & 3 & | & -3 \end{bmatrix}$$

Step 3 → Eliminate y from Eq^n ③ by using ②

$$R_3 \leftarrow R_3 - 3R_2$$

$$\left[\frac{A}{B}\right] = \begin{bmatrix} 2 & 3 & -1 & | & 5 \\ 0 & -2 & -1 & | & -7 \\ 0 & 0 & 6 & | & 18 \end{bmatrix}$$

Step 4 → Calculate the value of unknown variables (x... by using backward substitution:

for z → $6z = 18$

$z = 3$

for y → $-2y - z = -7$

$-2y - 3 = -7$

$y = 2$

for x → $2x + 3y - z = 5$

$2x + 6 - 3 = 5$

$x = 1$

Hence, the sol^n is

$x = 1, \; y = 2, \; z = 3$ Ans

Q → Represent the following system of Eq^n in matrix form.

$$2x + 3y + 4z = 9$$
$$3x + z = 8$$
$$4y + 9z = 7$$

~~Step 1 → First we write the given system of Eq^n into the~~

$$AX = B$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 3 & 0 & 1 \\ 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 7 \end{bmatrix}$$

$$[A/B] = \left[\begin{array}{ccc|c} 2 & 3 & 4 & 9 \\ 3 & 0 & 1 & 8 \\ 0 & 4 & 9 & 7 \end{array}\right] \quad \text{Ans}$$

Q → Solve the following system of linear Eq^n using Gauss Elimination method.

$$x + 4y - z = -5$$
$$x + y - 6z = -12$$
$$3x - y - z = 4$$

Step 1 → Write the given system of Eq^n into the $AX = B$ form

$$\begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -6 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -12 \\ 4 \end{bmatrix}$$

Now, we find $[A/B]$ form

$$[A/B] = \left[\begin{array}{ccc|c} 1 & 4 & -1 & -5 \\ 1 & 1 & -6 & -12 \\ 3 & -1 & -1 & 4 \end{array}\right] \begin{array}{l} - R_1 \\ - R_2 \\ - R_3 \end{array}$$

Step 2 → Eliminate x from Eq^n (2) & (3) by using Eq^n (1)

$$R_2 \leftarrow R_2 - R_1$$
$$R_3 \leftarrow R_3 - 3R_1$$

$$[A/B] = \left[\begin{array}{ccc|c} 1 & 4 & -1 & -5 \\ 0 & -3 & -5 & -7 \\ 0 & -13 & 2 & 19 \end{array}\right]$$

Step 3 → Eliminate y from Eq^n (3) by using Eq^n (2)

$$R_3 \leftarrow 3R_3 - 13R_2$$

$$\left[\frac{A}{B}\right] = \left[\begin{array}{ccc|c} 1 & 4 & -1 & -5 \\ 0 & -3 & -5 & -7 \\ 0 & 0 & 71 & 148 \end{array}\right]$$

Step 4 → Calculate the value of unknown variables $(x, y \,\&\, z)$

For z - $71z = 148$

$$z = \frac{148}{71}$$

$$z = 2.08$$

For y - $-3y - 5z = -7$

$$y = -1.14$$

For x - $x + 4y - z = -5$

$$x = 1.64$$

Hence, the sol^n is

$x = 1.64, \; y = -1.13, \; z = 2.08$ Ans

Q→ $5x - y - 3z = 142$

$x - 3y - z = -30$

$2x - y - 3z = -5$

Step 1→ Write the given system of eq^n into the $AX = B$

$$\begin{bmatrix} 5 & -1 & -3 \\ 1 & -3 & -1 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 142 \\ -30 \\ -5 \end{bmatrix}$$

Now we find $[A/B]$ form

$$[A/B] = \left[\begin{array}{ccc|c} 5 & -1 & -3 & 142 \\ 1 & -3 & -1 & -30 \\ 2 & -1 & -3 & -5 \end{array}\right] \begin{array}{l} -R_1 \\ -R_2 \\ -R_3 \end{array}$$

Step 2→ Eliminate x from eq^n ② & ③ by using eq^n ①

$R_2 \leftarrow 5R_2 - R_1$

$R_3 \leftarrow 5R_3 - 2R_1$

$$[A/B] = \left[\begin{array}{ccc|c} 5 & -1 & -3 & 142 \\ 0 & -14 & -2 & -292 \\ 0 & -3 & -9 & -309 \end{array}\right]$$

Step 3→ Eliminate y from eq^n ③ by using eq^n ②

$R_3 \leftarrow 14R_3 \cancel{+} -3R_1$

$$\left[\frac{A}{B}\right] = \left[\begin{array}{ccc|c} 5 & -1 & -3 & 142 \\ 0 & -14 & -2 & -292 \\ 0 & 0 & -120 & -3450 \end{array}\right]$$

Step 4→ Calculate the value of unknown variables $(x,$

for z - $\quad -120z = -3450$

$z = 28.75$

for y - $\quad -14y - 2z = -292$

$-14y - 2 \times 28.75 = -292$

$y = 16.75$

for x - $\quad 5x - \cancel{3}y - 3z = 142$

$5x - 16.75 - 3 \times 28.75 = 142$

$x = 49$

Hence, the sol^n is

$x = 49, \; y = 16.75, \; z = 28.75$ Ans

Q- $x + 0.01y - 0.02z = 3.3354$

$0.02x + y - 0.05z = 4.8241$

$0.03x - 0.01y + z = 7.3417$

Step ① → multiply all the Eq^n with 100

$$100x + y - 2z = 333.54$$
$$2x + 100y - 5z = 482.41$$
$$3x - y + 100z = 734.17$$

Step ② → write the given system of Eq^n into the $AX = B$ form

$$\begin{bmatrix} 100 & 1 & -2 \\ 2 & 100 & -5 \\ 3 & -1 & 100 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 333.54 \\ 482.41 \\ 734.17 \end{bmatrix}$$

Now we find $[\frac{A}{B}]$ form

$$\left[\frac{A}{B}\right] = \left[\begin{array}{ccc|c} 100 & 1 & -2 & 333.54 \\ 2 & 100 & -5 & 482.41 \\ 3 & -1 & 100 & 734.17 \end{array}\right] \begin{array}{l} — R_1 \\ — R_2 \\ — R_3 \end{array}$$

Step ③ → Eliminate x from Eq^n ② & ③ by using Eq^n ①

$R_2 \leftarrow 50R_2 - R_1$

$R_3 \leftarrow 100R_3 - 3R_1$

$$\left[\frac{A}{B}\right] = \left[\begin{array}{ccc|c} 100 & 1 & -2 & 333.54 \\ 0 & 4999 & -248 & 23786.96 \\ 0 & -103 & 10006 & 72416.38 \end{array}\right]$$

Step ④ → Eliminate y from Eq^n ③ by using Eq^n ②

$R_3 \leftarrow 4999R_3 + 103R_2$

$$\left[\frac{A}{B}\right] = \left[\begin{array}{ccc|c} 100 & 1 & -2 & 333.54 \\ 0 & 4999 & -248 & 23786.96 \\ 0 & 0 & 49994450 & 364459540.5 \end{array}\right]$$

Step ⑤ → Calculate the value of unknown variables $(x, y \,\&\, z)$

For z → $49994450\, z = 364459540.5$

$z = 7.29$

For y → $4999y - 248z = 23786.96$

$y = 5.12$

For x = $100x + y - 2z = 333.54$

$x = 3.43$

Hence, the sol^n is

$x = 3.43,\ y = 5.12,\ z = 7.29$ Ans

Q→ Solve the following system of Eq^n using partial pivoting technique (Gauss elimination method with pivoting)

$$2x+2y+z=6$$
$$4x+2y+3z=4$$
$$x+y+z=0$$

Step 1→ First, we arrange the given system of Eq^n into pivoting form

$$4x+2y+3z=4 \quad \text{—①}$$
$$2x+2y+z=6 \quad \text{—②}$$
$$x+y+z=0 \quad \text{—③}$$

Step 2→ Write the given system of Eq^n into the matrix

$$Ax = B$$

$$\begin{bmatrix} 4 & 2 & 3 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 0 \end{bmatrix}$$

Now we write $[A/B]$

$$\left[\frac{A}{B}\right] = \left[\begin{array}{ccc|c} 4 & 2 & 3 & 4 \\ 2 & 2 & 1 & 6 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

Step 3→ $R_2 \leftarrow 2R_2 - R_1$

$R_3 \leftarrow 4R_3 - R_1$

$$\left[\frac{A}{B}\right] = \left[\begin{array}{ccc|c} 4 & 2 & 3 & 4 \\ 0 & 2 & -1 & 8 \\ 0 & 2 & 1 & -4 \end{array}\right]$$

There is no need of pivoting because the given matrix in pivoting form.

Step 4→ $R_3 \leftarrow R_3 - R_2$

$$[A/B] = \left[\begin{array}{ccc|c} 4 & 2 & 3 & 4 \\ 0 & 2 & -1 & 8 \\ 0 & 0 & 2 & -12 \end{array}\right]$$

Step 5→ calculate the value of unknown variables (x, y & z)

for z - $\quad 2z = -12$

$$z = -6$$

for y. $\quad 2y - z = 8$

$$2y + 6 = 8$$
$$y = 1$$

for x - $\quad 4x + 2y + 3z = 4$

$$4x + 2 - 18 = 4$$
$$4x - 16 = 4$$

$x = 5$

Hence, the sol^n is

$x = 5, \ y = 1, \ z = -6$ Ans

Q→
$$x_1 + x_2 + x_3 + 4x_4 = -6$$
$$x_1 + x_2 + 6x_3 + x_4 = -5$$
$$5x_1 + x_2 + x_3 + x_4 = 4$$
$$x_1 + 7x_2 + x_3 + x_4 = 12$$

Step 1→ Pivoting form
$$5x_1 + x_2 + x_3 + x_4 = 4$$
$$x_1 + 7x_2 + x_3 + x_4 = 12$$
$$x_1 + x_2 + 6x_3 + x_4 = -5$$
$$x_1 + x_2 + x_3 + 4x_4 = -6$$

Step 2→ Matrix form
$$AX = B$$
$$\begin{bmatrix} 5 & 1 & 1 & 1 \\ 1 & 7 & 1 & 1 \\ 1 & 1 & 6 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ -5 \\ -6 \end{bmatrix}$$

Now we write $[A|B]$
$$\left[\frac{A}{B}\right] = \left[\begin{array}{cccc|c} 5 & 1 & 1 & 1 & 4 \\ 1 & 7 & 1 & 1 & 12 \\ 1 & 1 & 6 & 1 & -5 \\ 1 & 1 & 1 & 4 & -6 \end{array}\right]$$

Step 3→ Eliminate x_1 from Eq^n ② & ③ & ④ using Eq^n ①
$$R_2 \leftarrow 5R_2 - R_1$$
$$R_3 \leftarrow 5R_3 - R_1$$
$$R_4 \leftarrow 5R_4 - R_1$$
$$\left[\frac{A}{B}\right] = \left[\begin{array}{cccc|c} 5 & 1 & 1 & 1 & 4 \\ 0 & 34 & 4 & 4 & 56 \\ 0 & 4 & 29 & 4 & -29 \\ 0 & 4 & 4 & 19 & -34 \end{array}\right]$$

Step 4→ Eliminate x_2 from Eq^n ③ & ④ using ②
$$R_3 \leftarrow 34R_3 - 4R_2$$
$$R_4 \leftarrow 34R_4 - 4R_2$$
$$\left[\frac{A}{B}\right] = \left[\begin{array}{cccc|c} 5 & 1 & 1 & 1 & 4 \\ 0 & 34 & 4 & 4 & 56 \\ 0 & 0 & 970 & 120 & -1210 \\ 0 & 0 & 120 & \underline{630} & \underline{-1380} \end{array}\right]$$

Step 5 → Eliminate x_3 from Eq^n (4) using Eq^n (3)

$$R_4 \leftarrow 970R_4 - 120R_3$$

$$\left[\frac{A}{B}\right] = \left[\begin{array}{cccc|c} 5 & 1 & 1 & 1 & 4 \\ 0 & 34 & 4 & 4 & 56 \\ 0 & 0 & 970 & 120 & -1210 \\ 0 & 0 & 0 & 596700 & -1193400 \end{array}\right]$$

Step 6 → Calculate the value of unknown variables (x_1, ...

for x_4 -
$$596700x_4 = -1193400$$
$$x_4 = -2$$

for x_3 -
$$970x_3 + 120x_4 = -1210$$
$$970x_3 - 240 = -1210$$
$$x_3 = -1$$

for x_2 -
$$34x_2 + 4x_3 + 4x_4 = 56$$
$$34x_2 - 4 - 8 = 56$$
$$x_2 = 2$$

for x_1 -
$$5x_1 + x_2 + x_3 + x_4 = 4$$
$$5x_1 + 2 - 1 - 2 = 4$$
$$x_1 = 1$$

Hence, the sol^n is

$x_1 = 1$, $x_2 = 2$, $x_3 = -1$, $x_4 = -2$ Ans

Iterative methods for solving the system of linear Eq^n -

(a) Gauss Jacobi method

(b) Gauss siedal method

(a) Gauss Jacobi method →

Working Procedure →

Consider the Eq^n
$$a_1x + b_1y + c_1z = d_1 \quad —(1)$$
$$a_2x + b_2y + c_2z = d_2 \quad —(2)$$
$$a_3x + b_3y + c_3z = d_3 \quad —(3)$$

Step (1) → Arrange the Eq^n in pivoting form

Step (2) → calculate x, y & z

$$x = [d_1 - b_1y - c_1z]/a_1 \quad —(1)$$
$$y = [d_2 - a_2x - c_2z]/b_2 \quad —(2)$$
$$z = [d_3 - a_3x - b_3y]/c_3 \quad —(3)$$

Step ③ → Initial approximation

Put $x_0 = y_0 = z_0 = 0$

Step ④ → first approximation

$$x_1 = [d_1 - b_1 y_0 - c_1 z_0]/a_1$$
$$y_1 = [d_2 - a_2 x_0 - c_2 z_0]/b_2$$
$$z_1 = [d_3 - a_3 x_0 - b_3 y_0]/c_3$$

Second approximation

$$x_2 = [d_1 - b_1 y_1 - c_1 z_1]/a_1$$
$$y_2 = [d_2 - a_2 x_1 - c_2 z_1]/b_2$$
$$z_2 = [d_3 - a_3 x_1 - b_3 y_1]/c_3$$

$$\vdots$$

In general,

$$\left.\begin{aligned} x_{n+1} &= [d_1 - b_1 y_n - c_1 z_n]/a_1 \\ y_{n+1} &= [d_2 - a_2 x_n - c_2 z_n]/b_2 \\ z_{n+1} &= [d_3 - a_3 x_n - b_3 y_n]/c_3 \end{aligned}\right\} n = 0, 1, 2, 3, \ldots$$

Q→

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

Step ① → Calculate the value of x, y & z

$$x = [17 - y + 2z]/20 \quad \text{— ①}$$
$$y = [-18 - 3x + z]/20 \quad \text{— ②}$$
$$z = [25 - 2x + 3y]/20 \quad \text{— ③}$$

Step ② → First approximation → calculate x_1, y_1 & z_1 by using initial approximation $x_0 = y_0 = z_0 = 0$

$$x_1 = [17 - y_0 + 2z_0]/20 = 0.85$$
$$y_1 = [-18 - 3x_0 + z_0]/20 = -0.9$$
$$z_1 = [25 - 2x_0 + 3y_0]/20 = 1.25$$

Second approximation →

$$x_2 = 1.02$$
$$y_2 = -0.97$$

$z_2 = 1.03$

Third approximation →

$x_3 = 1.00$[redacted]

$y_3 = -1.00$[redacted]

$z_3 = 1.00$[redacted]

Fourth approximation →

$x_4 = 1.00$

$y_4 = -1.00$

$z_4 = 1.00$

Step ③ → Result → Since the value of last two approxim... same. Hence, the sol^n is

$x = 1,\ y = -1,\ z = 1$ Ans

Q→① $x + y + 5z = -1$

$2x + 4y = -12$

$5x - y + z = 10$

Q ② → $x + 4y + 2z = 15$

$x + 2y + 5z = 20$

$5x + 2y + z = 12$

Step ① → Calculate the value of x, y & z.

$x = [15 - 4y - 2z]/1$ — ①

$y = [20 - x - 5z]/2$ — ②

$z = [12 - 5x - 2y]/1$ — ③

Step ② → First approximation → Calculate x_1, y_1 & z_1 by us...

approximation $x_0 = y_0 = z_0 = 0$

$x_1 = [15 - 4y_0 - 2z_0] = 15$

$y_1 = [20 - x_0 - 5z_0]/2 =$ [redacted] 10

$z_1 = [12 - 5x_0 - 2y_0] = 12$

Second approximation →

$x_2 =$ [redacted] -49

$y_2 =$ [redacted] -27.50

$z_2 =$ [redacted] -83

same. Hence, the solⁿ is

$x = 1.22, \ y = -3.61, \ z = 0.28$ Ans

Q→
$$x + 4y + 2z = 15$$
$$x + 2y + 5z = 20$$
$$5x + 2y + z = 12$$

Step ① → Arrange the eqⁿ in pivoting form.

$$5x + 2y + z = 12$$
$$x + 4y + 2z = 15$$
$$x + 2y + 5z = 20$$

Step ② → calculate the value of x, y & z.

$$x = [12 - 2y - z]/5 \quad — ①$$
$$y = [15 - x - 2z]/4 \quad — ②$$
$$z = [20 - x - 2y]/5 \quad — ③$$

Step ③ → first approximation → calculate x_1, y_1, z_1 by using approximation $x_0 = y_0 = z_0 = 0$

$$x_1 = [12 - 2y_0 - z_0]/5 = 2.4$$
$$y_1 = [15 - x_0 - 2z_0]/4 = 3.75$$
$$z_1 = [20 - x_0 - 2y_0]/5 = 4$$

Second approximation →

$x_2 = 0.10$

$y_2 = 1.15$

$z_2 = 2.02$

Third approximation →

$x_3 = 1.54$

$y_3 = 2.72$

$z_3 = 3.52$

fourth approximation →

$x_4 = 0.61$

$y_4 = 1.61$

$z_4 = 2.60$

fifth approximation →

$x_5 = 1.24$

$y_5 = 2.30$

$z_5 = 3.23$

Sixth approximation →

$x_6 = 0.83$

$y_6 = 1.83$

$z_6 = 2.83$

Seventh approximation →

$x_7 = 1.10$

$y_7 = 2.13$

$z_7 = 3.10$

Eighth approximation →

$x_8 = 0.93$

$y_8 = 1.93$

$z_8 = 2.93$

Ninth approximation →

$x_9 = 1.04$

$y_9 = 2.05$

$z_9 = 3.04$

Tenth approximation →

$x_{10} = 0.97$

$y_{10} = 1.97$

$z_{10} = 2.97$

Eleventh approximation →

$x_{11} = 1.02$

$y_{11} = 2.02$

$z_{11} = 3.02$

Twelth approximation →

$x_{12} = 0.99$

$y_{12} = 1.99$

$z_{12} = 2.99$

Thirteeth approximation

$x_{13} = 1.01$

$y_{13} = 2.01$

$z_{13} = 3.01$

fourteeth approximation →

$x_{14} = 0.99$

$y_{14} = 1.99$

$z_{14} = 2.99$

fifteeth approximation →

$x_{15} = 1.01$

$y_{15} = 2.01$

$z_{15} = 3.01$

Step (4) → ~~Since~~, Result → Since, the value of last two app is continuous. Hence, the solⁿ is

$x = 1,\ y = 2,\ z = 3$ Ans

(b) Gauss Siedal Iterative method →

Working Procedure → Consider the system of eqⁿ

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Step(1) → First, we arrange the eqⁿ in pivoting form.

Step(2) → Calculate the value of unknown variables (x, y, z)

$$x = (d_1 - b_1y - c_1z)/a_1$$
$$y = (d_2 - a_2x - c_2z)/b_2$$
$$z = (d_3 - a_3x - b_3y)/c_3$$

Step (3) → First approximation →

$$x_1 = (d_1 - b_1y_0 - c_1z_0)/a_1$$
$$y_1 = (d_2 - a_2x_1 - c_2z_0)/b_2$$
$$z_1 = (d_3 - a_3x_1 - b_3y_1)/c_3$$

Second approximation →

$$x_2 = (d_1 - b_1y_1 - c_1z_1)/a_1$$
$$y_2 = (d_2 - a_2x_2 - c_2z_1)/b_2$$
$$z_2 = (d_3 - a_3x_2 - b_3y_2)/c_3$$

⋮

In general

$$\left.\begin{aligned} x_{n+1} &= (d_1 - b_1y_n - c_1z_n)/a_1 \\ y_{n+1} &= (d_2 - a_2x_{n+1} - c_2z_n)/b_2 \\ z_{n+1} &= (d_3 - a_3x_{n+1} - b_3y_{n+1})/c_3 \end{aligned}\right\}\ n = 0, 1, 2, \ldots$$

Q→ $2x - 3y + 20z = 25$

$3x + 20y - z = -18$

$20x + y - 2z = 17$

Step (1)→ $20x + y - 2z = 17$ (Pivoting)

$3x + 20y - z = -18$

$2x - 3y + 20z = 25$

Step (2)→ calculate the value of x, y & z

$x = (17 - y + 2z)/20$ — ①

$y = (-18 - 3x + z)/20$ — ②

$z = (25 - 2x + 3y)/20$ — ③

Step (3)→ First approximation→

$x_1 = (17 - y_0 + 2z_0)/20 = 0.9$

$y_1 = (-18 - 3x_1 + z_0)/20 = -1.0$

$z_1 = (25 - 2x_1 + 3y_1)/20 = 1.0$

Second approximation→

$x_2 = (17 - y_1 + 2z_1)/20 = 1.0$

$y_2 = (-18 - 3x_2 + z_1)/20 = -1.0$

$z_2 = (25 - 2x_2 + 3y_2)/20 = 1.0$

Third approximation→

$x_3 = (17 - y_2 + 2z_2)/20 = 1.0$

$y_3 = (-18 - 3x_3 + z_2)/20 = -1.0$

$z_3 = (25 - 2x_3 + 3y_3)/20 = 1.0$

Result→ since the value of last 2 approximation are same.
Hence, the solⁿ is
$x = 1.0$, $y = -1.0$, $z = 1.0$ Ans

ILL condition→

The linear system is known as ill conditioned ~~of~~ if small changes in the coefficient of the eqⁿ result in large changes in the value of unknown variable.

Q→ Establish whether the system

$$1.01x + 2y = 2.01$$
$$x + 2y = 2$$

is well conditioned or ill conditioned

On solving we get

$x = 1$

$y = 0.5$

After changing

$$x + 2.01y = 2.01$$
$$x + 2y = 2$$

on solving we get

$x = 0$

$y = 1$

Because the changes in the values are large. Hence, the system is ill conditioned.

Numerical integration→ Unit→3

$$\int_a^b f(x)dx = [F(x)]_a^b + c$$
$$= F(b) - F(a)$$

where→

a→ lower limit

b→ upper limit

c→ constent of integration

(I) Trapezoidal rule→

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})\right]$$

where

a→ lower limit

b→ upper limit

h→ difference b/w two consecutive values

Relation b/w n & h→

$$h = \frac{b-a}{n}$$

where n is no. of subinterval

Note→ No. of subinterval is one less than the no. of terms.

(2) Simpson's $\frac{1}{3}$ Rule →

$$\int_a^b f(x)dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\cdots)+2(y_2+y_4+y_6+\cdots)\right]$$

(3) Simpson's $\frac{3}{8}$ Rule →

$$\int_a^b f(x)dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots)+2(y_3+y_6+y_9+\cdots)\right]$$

Remark → The above formula is applicable only when the below condition is true.

(a) Trapezoidal rule → $\frac{\text{No. of subinterval}}{1}$

(b) Simpson's $\frac{1}{3}$ rule → $\frac{\text{No. of subinterval}}{2}$

(c) Simpson's $\frac{3}{8}$ rule → $\frac{\text{No. of subinterval}}{3}$

Q → Evaluate $\int_0^6 \frac{1}{1+x}dx$

(a) Trapezoidal Rule

(b) Simpson's $\frac{1}{3}$ Rule

(c) Simpson's $\frac{3}{8}$ Rule

$a=0$

$b=6$

Let the no. of subinterval $(n)=6$

$$h = \frac{b-a}{n} = \frac{6-0}{6} = 1$$

$\boxed{h=1}$

Now, we make the table

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.33	0.25	0.20	0.17	0.14

(a) Apply the Trapezoidal rule

$$\int_a^b f(x)dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots)\right]$$

$$\int_0^6 \frac{1}{1+x}dx = \frac{1}{2}\left[(1+0.14)+2(0.5+0.33+0.25+0.20+0.1\ldots\right.$$

$$= 0.5[1.14+2(1.45)]$$

$$= 2.02 \text{ Ans}$$

(b) Apply Simpson's $\frac{1}{3}$ rule

$$\int_a^b f(x)dx = \frac{h}{3}\left[\underline{(y_0+y_n)}+4\underline{(y_1+y_3+y_5+\cdots)}+2\underline{(y_2+\ldots}\right.$$

$$= \frac{1}{3}\left[(1+0.14)+4(0.5+0.25+0.17)+2(0.33+0\ldots\right.$$

$$= \frac{1}{3}\left[(1.14)+ 3.68 + 1.06\right]$$

$$= 1.96 \text{ Ans}$$

(c) Apply Simpson's $\frac{3}{8}$ rule

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+\cdots)+2(y_3+y_6+\cdots\right.$$

$$= 3\times\frac{1}{8}\left[(1+0.14)+3(0.5+0.33+0.20+0.17)+2\ldots\right.$$

$$= \frac{3}{8}[1.14+3.60+0.50]$$

$$= 1.97 \text{ Ans}$$

Q→ Evaluate

$\int_0^{\pi/2} \sin x\,dx$ by using Simpson's $\frac{1}{3}$ rule using 11 ...

$a = 0$

$b = \pi/2$

$h = \frac{b-a}{n} = \frac{\frac{\pi}{2}-0}{10} = \frac{\pi}{20}$

$f(x) = \sin x$

~~Now, we make the table~~

~~x~~	~~0~~	~~$\frac{\pi}{20}$~~	~~$\frac{\pi}{10}$~~	~~$\frac{3\pi}{20}$~~	~~$\frac{\pi}{5}$~~	~~$\frac{\pi}{4}$~~	~~$\frac{3\pi}{10}$~~	~~$\frac{\pi}{2}$~~
~~f(x)~~	~~0~~	~~0.16~~	~~0.31~~	~~0.45~~	~~0.59~~	~~0.71~~	~~0.81~~	~~1~~

Now, we make the table

x	0	$\frac{\pi}{20}$	$\frac{\pi}{10}$	$\frac{3\pi}{20}$	$\frac{\pi}{5}$	$\frac{\pi}{4}$	$\frac{3\pi}{10}$	$\frac{7\pi}{20}$	$\frac{2\pi}{5}$	...
$f(x)$	0	0.0027	0.0055	0.0082	0.0110	0.0137	0.0164	0.0192	0.0219	...

$$\int_0^{\pi/2} \sin\{x\} = \frac{h}{3}\left[(y_0+y_{10})+4(y_1+y_3+y_5+y_7+y_9)+2(y_2+y_4+y_6\ldots\right.$$

$$= \frac{\pi}{60}\left[(0+0.0274)+4(0.0027+0.0082+0.0137+0.0192+0.024\ldots\right.$$

$$+2(0.0055+0.0110+0.0164+0.0219)]$$

$$= \frac{\pi}{60}[0.0274+0.2740+0.1096)$$

$$= \frac{\pi}{60}(0.4110) = 0.0215 \text{ Ans}$$

(4) <u>Boole's formula →</u>

$$\int_a^b f(x)\,dx = \frac{2h}{45}\left[7(y_0+y_n)+32(y_1+y_3+y_5)+12(y_2+y_6+\cdots)+14(y_4+y_8+\cdots)\right]$$

No. of subinterval/4

(5) <u>Weddle's rule →</u>

$$\int_a^b f(x)\,dx = \frac{3h}{10}\left[(y_0+y_2+\cdots+y_n)+5(y_1+y_5+\cdots)+6(y_3+y_9+\cdots)+2(y_6+y_{12}+\cdots)\right]$$

No. of subinterval/6

Q→ Evaluate $\int_1^2 e^{-\sqrt{x}/2}\,dx$

Let the no. of subinterval $n=6$

$a=1$

$b=2$

$h = \frac{b-a}{n} = \frac{2-1}{6} = 0.17$

Now, we make the table

$f(x) = e^{-\sqrt{x}/2}$

x	1	1.17	1.34	1.51	1.68	1.85	2
$f(x)$	0.6065	0.5823	0.5606	0.5410	0.5231	0.5066	0.4931

Apply the Weddle's rule

$$\int_a^b f(x)\,dx = \frac{3h}{10}\left[(y_0+y_2+y_4)+5(y_1+y_5+\cdots)+6(y_3+y_9+\cdots)+2(y_6+y_{12}+\cdots)\right]$$

$$= \frac{3\times0.17}{10}\left[(0.6065+0.5606+0.5231)+5(0.5823+0.5066)+6(0.5410+\cdots)+2(0.4931)\right]$$

$$= \frac{3\times0.17}{10}(1.6902+5.4445+3.2460+0.9862)$$

$$= 0.0510 * 11.3669 = 0.5797 \text{ Ans}$$

Q→ compute the value of $I = \int_{0.2}^{1.5} e^{-x^2} dx$ using 4 subinter

Q→ Using simpson's $\frac{3}{8}$ rule to evaluate

$$\int_0^6 \frac{dx}{1+x^4}$$

$a = 0$

$b = 6$

$n = 6$

$h = \frac{6-0}{6} = \frac{6}{6} = 1$

Now, we make the table

x	0	1	2	3	4	5	6
$f(x)$	1	0.5000	0.0588	0.0122	0.0039	0.0016	0.000

$$\int_0^6 \frac{dx}{1+x^4} = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right]$$

$$= \frac{3}{8}\left[(1+0.0008)+3(0.5000+0.0588+0.0039+0.0016)+2\right.$$

$$= \frac{3}{8}(1.0008+1.6929+0.0244)$$

$$= 1.0193 \text{ Ans}$$

Q→ $I = \int_{0.2}^{1.5} e^{-x^2} dx$

$b = 1.5$

$a = 0.2$

$n = 4$

$h = \frac{1.5-0.2}{4} = 0.3$

Now, we make the table

x	0.2	0.5	0.8	1.1	1.4
$f(x)$	0.9608	0.7788	0.5273	0.2982	0.1409

Apply simpson's $\frac{1}{3}$ rule

$$\int_{0.2}^{1.5} e^{-x^2} = \frac{h}{3}\left[(y_0+y_4)+4(y_1+y_3)+2(y_2)\right]$$

$$= \frac{0.3}{3}\left[(0.9608+0.1409)+4(0.7788+0.2982)+2(0.5273)\right.$$

$$= \frac{0.3}{3}\left(1.1017 + 4.3080 + 1.0546\right)$$

$$= 1.8977 \text{ Ans}$$

Q- Use Simpson's rule for evaluating $\int_{-0.6}^{0.3} f(x)dx$ from the table given below-

x	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3
f(x)	4	2	5	3	-2	1	6	4	2	8

$a = -0.6$
$b = 0.3$
$n = 9$

$$h = \frac{0.3+0.6}{9} = \frac{0.9}{9} = 0.1000$$

Apply Simpson's $\frac{3}{8}$ rule

$$\int_{-0.6}^{0.3} f(x)dx = \frac{3h}{8}\left[\left(y_0 + y_9\right) + 3\left(y_1 + y_2 + y_4 + y_5 + y_7 + y_8\right) + 2\left(y_3 + y_6\right)\right]$$

$$= \frac{3 \times 0.1000}{8}\left[\left(4+8\right) + 3\left(2+5+(-2)+1+4+2\right) + 2\left(3+6\right)\right]$$

$$= \frac{0.3000}{8}\left[12 + 36 + 18\right)$$

$$= 2.4750 \text{ Ans}$$

Q→ A train is moving at the speed of 30 m/sec. Suddenly breaks are apply the speed of the train per second after t sec. is given by -

Time(t)	0	5	10	15	20	25	30	35	40	45
Speed(v)	30	24	19	16	13	11	10	8	7	5

Apply Simpson's rule to determine the distance move by the train in 45 sec.

$$\int_0^{45} v \cdot dt$$

$a = 0$
$b = 45$
$n = 9$

$$h = \frac{45-0}{9} = \frac{45}{9} = 5$$

$$\int_0^{45} u \cdot dt = \frac{3h}{8}\Big[(y_0+y_9)+3(y_1+y_2+y_4+y_5+y_7+y_8)+2(y_3+y_6\Big]$$

$$= \frac{15}{8}\Big[(30+5)+3(24+19+13+11+8+7)+2(16+10)\Big]$$

$$= \frac{15}{8}(35+246+52)$$

$$= 624.3750 \text{ Ans}$$

Numerical differentiation:-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

We know that the Newton's forward diff. formula is:-

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

where

$$u = \frac{x-x_0}{h} \quad \text{—— (2)}$$

Differentiate eq^n (1) w. r. to u.

$$\frac{dy}{du} = \Delta y_0 + \frac{(2u-1)}{2}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{6}\Delta^3 y_0 + \cdots \quad \text{—— (3)}$$

Differentiate eq^n (2) w.r. to x

$$\frac{du}{dx} = \frac{1}{h}(1-0) = \frac{1}{h} \quad \text{—— (4)}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{h}\Big[\Delta y_0 + \frac{(2u-1)}{2}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{6}\Delta^3 y_0 + \cdots\Big]$$

Q:- Find $\frac{dy}{dx}$ at $x = 1895$ from the following -

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	46				
		20			
1901	66				

1911	81				
		15			
			-3		
				2	
		12			-4
1921	93			-2	
			-5		
		7			
1931	100				

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{6}\Delta^3 y_0 + \text{----}\right]$$

$$u = \frac{x-x_0}{h} = \frac{1895-1891}{10} = \frac{4}{10} = .4$$

$$\left(\frac{dy}{dx}\right)_{x=1895} = \frac{1}{10}\left[20 + \frac{(2\times .4-1)}{2}\times(-5) + \frac{(3(.4)^2-6\times .4+2)\times 2}{6}\right]$$

$$= \frac{1}{10}\left[20 + 0.5000 + 0.0267\right]$$

$$= 2.0527 \text{ Ans}$$

(6) Euler's maclaurine formulla →

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\text{---})\right] - \frac{h^2}{12}\left(y_n' - y_0'\right) + \frac{h^4}{720}\left(y_n''' - y_0'''\right)$$

— ①

Q → Use Euler maclaurine formulla to evaluate $\int_0^1 \frac{1}{1+x}\,dx$

Given → Lower limit (a) = 0

upper limit (b) = 1

$$f(x) = \frac{1}{1+x}$$

Let the no of subintervual (n) = 10

$$h = \frac{b-a}{n} = \frac{1-0}{10} = \frac{1}{10} = .1$$

Now, we make the table

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$f(x)$	1	0.9091	0.8333	.7692	.7143	.6667	.6250	.5882	.5556	.5263	.5000
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_n

Apply maclaurine formulla

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\text{---})\right] - \frac{h^2}{12}\left(y_n' - y_0'\right) + \frac{h^4}{720}\left(y_n''' - y_0'''\right)$$

$$y = \frac{1}{1+x} = (1+x)^{-1}$$

$$y' = -1(1+x)^{-1-1} = -(1+x)^{-2} = \frac{-1}{(1+x)^2}$$

$$y_n' = y_1' = \frac{-1}{(1+x)^2} = \frac{-1}{(1+1)^2} = \frac{-1}{4}$$

$$y_0' = y_0' = \frac{-1}{(1+0)^2} = -1$$

$$y' = \frac{-1}{(1+x)^2} = -(1+x)^{-2}$$

$$y'' = 2(1+x)^{-3}$$

$$y''' = -6(1+x)^{-4} = \frac{-6}{(1+x)^4}$$

$$y_n''' = y_1''' = \frac{-6}{(2)^4} = \frac{-6}{16} = \frac{-3}{8}$$

$$y_0''' = y_0''' = \frac{-6}{1} = -6$$

$$\int_0^1 \frac{1}{1+x}\,dx = \frac{0.1}{2}\Big[(1+0.5000) + 2(0.9091 + 0.8333 + 0.7692 + 0.6667 + 0.6250 + 0.5882 + 0.5556 + 0.5263)\Big] - \frac{(0.1)^2}{12}\Big(\frac{-1}{4} \ldots$$

$$+ \frac{(0.1)^4}{720}\left(\frac{-3}{8} + 6\right)$$

$$= 0.6932 \text{ Ans}$$

Q → $\boxed{\frac{1}{51^2} + \frac{1}{53^2} + \frac{1}{55^2} + \cdots + \frac{1}{99^2}}$

$y_0 \quad y_1 \quad\quad y_n$

Let $f(x) = \frac{1}{x^2}$

$$h = \frac{b-a}{n}$$

$a = 51$

$b = 99$

$h = 2$

$$n = \frac{b-a}{h} = \frac{99-\ldots}{2}$$

$$\int_a^b f(x) = \frac{h}{2}\Big[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots)\Big] - \frac{h^2}{12}\left(y_n' - \ldots\right.$$

$$+ \frac{h^4}{720}\left(y_n''' - y_0'''\right)$$

$$y = \frac{1}{x^2} = x^{-2}$$

$$y' = -2x^{-3} = \frac{-2}{x^3}$$

$$y_n' = y_{99}' = \frac{-2}{99^3}$$

$$y_0' = y_{51}' = \frac{-2}{51^3}$$

$$y' = \frac{-2}{x^3} = -2x^{-3}$$

$$y'' = 6x^{-4}$$

$$y''' = -24x^{-5} = \frac{-24}{x^5}$$

$$y_n''' = y_{99}''' = \frac{-24}{99^5}$$

$$y_0''' = y_{51}''' = \frac{-24}{51^5}$$

$$\int_{51}^{99} \frac{1}{x^2}dx = \frac{2}{2}\left[\left(\frac{1}{51^2}+\frac{1}{99^2}\right)+2\left(\frac{1}{53^2}+\frac{1}{55^2}+\cdots+\frac{1}{97^2}\right)\right] - \frac{4}{12}\left(\frac{-2}{99^3}+\frac{2}{51^3}\right)$$

$$+\frac{16}{720}\left(\frac{-24}{99^5}+\frac{24}{51^5}\right)$$

$$= \left[\left(\frac{1}{51^2}+\frac{1}{99^2}\right)+2\left(\frac{1}{53^2}+\frac{1}{55^2}+\cdots+\frac{1}{97^2}\right)\right]+\frac{2}{3}\left(\frac{1}{99^3}-\frac{1}{51^3}\right)$$

$$-\frac{8}{15}\left(\frac{1}{99^5}-\frac{1}{51^5}\right)$$

$$\int_{51}^{99}\frac{1}{x^2}dx - \frac{2}{3}\left(\frac{1}{99^3}-\frac{1}{51^3}\right)+\frac{8}{15}\left(\frac{1}{99^5}-\frac{1}{51^5}\right) = \left(\frac{1}{51^2}+\frac{1}{99^2}\right)$$

$$+2\left(\frac{1}{53^2}+\frac{1}{55^2}+\cdots\frac{1}{97^2}\right)$$

$$+\left(\frac{1}{51^2}+\frac{1}{99^2}\right)-\left(\frac{1}{51^2}+\frac{1}{99^2}\right)$$

$$\left[\frac{x^{-1}}{-1}\right]_{51}^{99} - \frac{2}{3}\left(\frac{1}{99^3}-\frac{1}{51^3}\right)+\frac{8}{15}\left(\frac{1}{99^5}-\frac{1}{51^5}\right) = 2\left(\frac{1}{51^2}+\frac{1}{99^2}\right)+2$$

$$\left(\frac{1}{53^2}+\frac{1}{55^2}+\cdots\frac{1}{97^2}\right)-\left(\frac{1}{51^2}+\frac{1}{99^2}\right)$$

$$\left[\frac{-1}{x}\right]_{51}^{99} - \frac{2}{3}\left(\frac{1}{99^3}-\frac{1}{51^3}\right)+\frac{8}{15}\left(\frac{1}{99^5}-\frac{1}{51^5}\right)+\left(\frac{1}{51^2}+\frac{1}{99^2}\right) = 2$$

$$\left(\frac{1}{51^2}+\frac{1}{53^2}+\cdots\frac{1}{99^2}\right)$$

$$\left[-\frac{1}{99}-\left(-\frac{1}{51}\right)\right]-\frac{2}{3}\left(\frac{1}{99^3}-\frac{1}{51^3}\right)+\frac{8}{15}\left(\frac{1}{99^5}-\frac{1}{51^5}\right)+\left(\frac{1}{51^2}+\frac{1}{99^2}\right) = 2$$

$$\left(\frac{1}{51^2}+\frac{1}{53^2}+\cdots\frac{1}{99^2}\right)$$

$$\left(\frac{1}{51}-\frac{1}{99}\right)-\overset{0.6667}{\cancel{\frac{2}{3}}}\left(\frac{1}{99^3}-\frac{1}{51^3}\right)+\overset{0.5333}{\cancel{\frac{8}{15}}}\left(\frac{1}{99^5}-\frac{1}{51^5}\right)+\left(\frac{1}{51^2}+\frac{1}{99^2}\right) = 2$$

$$\left(\frac{1}{51^2}+\frac{1}{53^2}+\cdots\frac{1}{99^2}\right)$$

$$\frac{1}{51^2}+\frac{1}{53^2}+\cdots\frac{1}{99^2} = 0.004999 \text{ Ans}$$

Q → Evaluate by Euler's maclaurine formulla

$$\frac{1}{201^2}+\frac{1}{203^2}+\frac{1}{205^2}+\cdots\frac{1}{299^2}$$

Let $f(x) = \frac{1}{x^2}$

$a = 201$

$b = 299$

$h = 2$

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots+y_{n-1})\right] - \frac{h^2}{12}\ldots + \frac{h^4}{720}\left(y_n''' - y_0'''\right)$$

$$y = \frac{1}{x^2} = x^{-2}$$

$$y' = -2x^{-3} = \frac{-2}{x^3}$$

$$y_n' = y_{299}' = \frac{-2}{299^3}$$

$$y_0' = y_{201}' = \frac{-2}{201^3}$$

$$y' = \frac{-2}{x^3} = -2x^{-3}$$

$$y'' = 6x^{-4}$$

$$y''' = -24x^{-5} = \frac{-24}{x^5}$$

$$y_n''' = y_{299}''' = \frac{-24}{299^5}$$

$$y_0''' = y_{201}''' = \frac{-24}{201^5}$$

$$\int_{201}^{299}\frac{1}{x^2}dx = \frac{2}{2}\left[\left(\frac{1}{201^2}+\frac{1}{299^2}\right)+2\left(\frac{1}{203^2}+\frac{1}{205^2}+\cdots\frac{1}{297^2}\right)\right] - \frac{4}{12}\ldots$$

$$+\frac{16}{720}\left(\frac{-24}{299^5}+\frac{24}{201^5}\right)$$

$$= \left[\left(\frac{1}{201^2}+\frac{1}{299^2}\right)+2\left(\frac{1}{203^2}+\frac{1}{205^2}+\cdots+\frac{1}{297^2}\right)\right] + \frac{2}{3}\left(\frac{1}{299^3}-\ldots\right.$$

$$-\frac{8}{15}\left(\frac{1}{299^5}-\frac{1}{201^5}\right)$$

$$\int_{201}^{299}\frac{1}{x^2}dx - \frac{2}{3}\left(\frac{1}{299^3}-\frac{1}{201^3}\right)+\frac{8}{15}\left(\frac{1}{299^5}-\frac{1}{201^5}\right) = \left(\frac{1}{201^2}+\frac{1}{299^2}\right)$$

$$+2\left(\frac{1}{203^2}+\frac{1}{205^2}+\cdots+\frac{1}{297^2}\right)+\left(\frac{1}{201^2}+\frac{1}{299^2}\right)$$

$$-\left(\frac{1}{201^2}+\frac{1}{299^2}\right)$$

$$\left[\frac{x^{-1}}{-1}\right]_{201}^{299} - \frac{2}{3}\left(\frac{1}{299^3}-\frac{1}{201^3}\right)+\frac{8}{15}\left(\frac{1}{299^5}-\frac{1}{201^5}\right) = 2\left(\frac{1}{201^2}+\frac{1}{299^2}\right)$$

$$+2\left(\frac{1}{203^2}+\frac{1}{205^2}+\cdots+\frac{1}{297^2}\right)-\left(\frac{1}{201^2}+\frac{1}{299^2}\right)$$

$$\left[\frac{-1}{x}\right]_{201}^{299} - \frac{2}{3}\left(\frac{1}{299^3}-\frac{1}{201^3}\right)+\frac{8}{15}\left(\frac{1}{299^5}-\frac{1}{201^5}\right)+\left(\frac{1}{201^2}+\frac{1}{299^2}\right)$$

$$= 2\left(\frac{1}{201^2}+\frac{1}{203^2}+\cdots+\frac{1}{299^2}\right)$$

$$\left[-\frac{1}{299}-\left(\frac{-1}{201}\right)\right] - \frac{2}{3}\left(\frac{1}{299^3}-\frac{1}{201^3}\right)+\frac{8}{15}\left(\frac{1}{299^5}-\frac{1}{201^5}\right)+\left(\frac{1}{201^2}+\frac{1}{299^2}\right)$$

$$= 2\left(\frac{1}{201^2}+\frac{1}{203^2}+\cdots+\frac{1}{299^2}\right)$$

$$\left(\frac{1}{201}-\frac{1}{299}\right)-0.6667\left(\frac{1}{299^3}-\frac{1}{201^3}\right)+0.5333\left(\frac{1}{299^5}-\frac{1}{201^5}\right)+\left(\frac{1}{201^2}+\frac{1}{299^2}\right)$$

$$= 2\left(\frac{1}{201^2}+\frac{1}{203^2}+\cdots+\frac{1}{299^2}\right)$$

$$\frac{1}{201^2}+\frac{1}{203^2}+\cdots+\frac{1}{299^2} = 0.000833 \text{ Ans}$$

Numerical Differentiation:

We know that N.F. diff formulla

$$y = y_0 + u\Delta y_0 + \frac{(u^2-u)}{2!}\Delta^2 y_0 + \frac{(u^3-3u^2+2)}{3!}\Delta^3 y_0 + \frac{(u^4-6u^3+9u^2+2u-6)}{4!}\Delta^4 y_0 + \cdots \quad (1)$$

$$u = \frac{x-x_0}{h} \quad (2)$$

$$\frac{dy}{du} = 0 + \Delta y_0 + \frac{(2u-1)}{2}\Delta^2 y_0 + \frac{(3u^2-6u)}{6}\Delta^3 y_0 + \frac{(4u^3-18u^2+18u+2)}{24}\Delta^4 y_0 + \cdots$$

$$\frac{du}{dx} = \frac{1}{h}$$

$$\frac{dy}{dx} = \frac{du}{dx}\cdot\frac{dy}{du} = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2}\Delta^2 y_0 + \frac{(3u^2-6u)}{6}\Delta^3 y_0 + \frac{(4u^3-18u^2+18u+2)}{24}\Delta^4 y_0 + \cdots\right.$$

$$\frac{d^2y}{dx^2} = \frac{d}{du}\left[\frac{dy}{dx}\right]\frac{du}{dx}$$

$$= \frac{d}{du}\left[\frac{1}{h}\Delta y_0 + \frac{(2u-1)}{2}\Delta^2 y_0 + \frac{(3u^2-6u)}{6}\Delta^3 y_0 + \frac{(4u^3-18u^2+18u-\ldots}{24}\right.$$

$$= \frac{1}{h^2}\frac{d}{du}\left(\Delta y_0 + \frac{(2u-1)}{2}\Delta^2 y_0 + \frac{(3u^2-6u)}{6}\Delta^3 y_0 + \frac{(4u^3-18u^2+18u-\ldots}{24}\right.$$

$$= \frac{1}{h^2}\left(0 + \frac{2}{2}\Delta^2 y_0 + \frac{(6u-6)}{6}\Delta^3 y_0 + \frac{12u^2-36u+18}{24}\Delta^4 y_0 + \cdots\right)$$

$$= \frac{1}{h^2}\left(\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{(2u^2-6u+3)}{4}\Delta^4 y_0 + \cdots\right)$$

Q → Find $f'(1.1)$ & $f''(1.1)$ from the following table.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1.0	0.0000	0.1280	0.2980	0.0180	0.0600
1.2	0.1280	0.4260	0.3160	0.0780	-0.0400
1.4	0.5540	0.7420	0.3940	0.0380	
1.6	1.2960	1.1360	0.4320		
1.8	2.4320	1.5680			
2.0	4.0000				

$$u = \frac{x-x_0}{h} = \frac{1.1-1.0}{.2} = 0.5$$

$$\frac{dy}{dx} = \frac{1}{h}\left(\Delta y_0 + \frac{(2u-1)}{2}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{6}\Delta^3 y_0 + \frac{(4u^3-18u^2+18u-\ldots}{24}\right.$$

$$= \frac{1}{.2}(0.1280 + (0) + 0.2083 \times 0.0180 + 0.2917 \times 0.0600)$$

$$= \frac{1}{.2}(0.1280 + 0.0037 + 0.0175)$$

$$= \frac{1}{.2}(0.1492) = 0.746 \text{ Ans}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left(\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{(2u^2-6u+3)}{4}\Delta^4 y_0 + \cdots\right)$$

$$= \frac{1}{.04}(0.2980 + (-0.5)0.0180 + 0.1250 \times 0.0600)$$

$$= \frac{1}{.04}(0.2980 - 0.0090 + 0.0075)$$

$$= \cancel{0.298}$$

$$= \frac{1}{.04}(0.2965) = 7.4125 \text{ Ans}$$

Q→

t(sec)	0	0.2	0.4	0.6	0.8	1.0	1.2
Q(Rad)	0	0.12	0.49	1.12	2.02	3.20	4.67

Calculate the angular velocity & angular acceleration of the road at $t = 0.6$ sec.

t	Q	ΔQ	$\Delta^2 Q$	$\Delta^3 Q$	$\Delta^4 Q$
0	0	0.12	.25	0.01	0
0.2	0.12	0.37	.26	.01	0
0.4	0.49	0.63	.27	.01	0
0.6	1.12	0.90	.28	.01	
0.8	2.02	1.18	.29		
1.0	3.20	1.47			
1.2	4.67				

$$\frac{dy}{dx} = \Delta y_0 + \frac{(2u-1)}{2}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{6}\Delta^3 y_0 + \cdots$$

$$= .90 - \frac{1}{2}\times .28 + \frac{1}{3}\times .01$$

$$= 0.76 \text{ Ans}$$

$$\frac{d^2y}{dx^2} = \Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{(2u^2-6u+3)}{4}\Delta^4 y_0 + \cdots$$

$$= .28 + (-1)\times .01$$

$$= 0.27 \text{ Ans}$$

Solution of differential eqn →

Std. form of differential eqn →

$$\boxed{\frac{dy}{dx} = f(x,y)}$$

for ex → $\frac{dy}{dx} + 2 - x^2 = 0$

$$\boxed{\frac{dy}{dx} = x^2 - 2}$$

⇒ x is the independent variable.

(1) Picard's method→

suppose $\frac{dy}{dx} = f(x, y)$

$$dy = f(x, y)\,dx$$

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\,dx$$

Integration with respect to x & y

$$[y]_{y_0}^{y} = \int_{x_0}^{x} f(x, y)\,dx$$

$$y - y_0 = \int_{x_0}^{x} f(x, y)\,dx$$

$$y = y_0 + \int_{x_0}^{x} f(x, y)\,dx$$

~~first~~ approximation→

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$\vdots$$

$$\boxed{y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\,dx} \quad n = 0, 1, 2, ---$$

Q→ Use Picard method to calculate y at x = 0·2 with condition $\frac{dy}{dx} = x - y$ and $y = 1$ at $x = 0$.

$$\frac{dy}{dx} = x - y$$

$$\boxed{f(x, y) = x - y}$$

initial condition→

$$\left.\begin{array}{l} x_0 = 0 \\ y_0 = 1 \end{array}\right\}$$

$y = ?$ at $x = 0·2$

Apply the Picard method

$$\boxed{y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\,dx} \quad n = 0, 1, 2, ---$$

First approximation:

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$= 1 + \int_0^x (x - y_0)\,dx$$

$$= 1 + \int_0^x (x-1)\,dx$$

$$= 1 + \left[\frac{x^2}{2} - x\right]_0^x$$

$$\boxed{y_1 = 1 + \frac{x^2}{2} - x}$$

$y_1(0.2) = 0.8200$
= 0.5162
= 0.6170
= .5733
= .5910
= .5836
= .5867
= .5854
= .5859
= .5857
= .5858
= .5858

Second approximation:

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$= 1 + \int_{x_0}^{x} f\left(x, 1 + \frac{x^2}{2} - x\right)dx$$

$$= 1 + \int_{x_0}^{x} x - \left(1 + \frac{x^2}{2} - x\right)dx$$

$$= 1 + \int_{x_0}^{x} \left(x - 1 - \frac{x^2}{2} + x\right)dx$$

$$= 1 + \left[\frac{x^2}{2} - x - \frac{x^3}{6} + \frac{x^2}{2}\right]_0^x$$

$$\boxed{y_2 = 1 + x^2 - x - \frac{x^3}{6}}$$

$y_2(0.2) = 0.8387$

(1) Pi[illegible] approximation→

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx$$

$$= 1 + \int_{x_0}^{x} f\left(x, 1+x^2-x-\frac{x^3}{6}\right)dx$$

$$= 1 + \int_{x_0}^{x} x-\left(1+x^2-x-\frac{x^3}{6}\right)dx$$

$$= 1 + \int_{x_0}^{x} \left(x-1-x^2+x+\frac{x^3}{6}\right)dx$$

$$= 1 + \left[\frac{x^2}{2}-x-\frac{x^3}{3}+\frac{x^2}{2}+\frac{x^4}{24}\right]_0^x$$

$$\boxed{y_3 = 1+x^2-x-\frac{x^3}{3}+\frac{x^4}{24}}$$

$$y_3(0.2) = 0.8374$$

Fourth approximation→

$$y_4 = y_0 + \int_{x_0}^{x} f(x, y_3)\,dx$$

$$= 1 + \int_{x_0}^{x} f\left(x, 1+x^2-x-\frac{x^3}{3}+\frac{x^4}{24}\right)dx$$

$$= 1 + \int_{x_0}^{x} \left(x-1-x^2+x+\frac{x^3}{3}-\frac{x^4}{24}\right)dx$$

$$\boxed{y_4 = 1+\frac{x^2}{2}-x-\frac{x^3}{3}+\frac{x^2}{2}+\frac{x^4}{12}-\frac{x^5}{120}}$$

$$y_4(0.2) = 0.8374$$

Since, the value of last 2 approximation are same. Hence of given D.E. is

$$\boxed{y = 0.8374 \text{ at } x = 0.2}$$ Ans

Q→ If $\frac{dy}{dx} = \frac{y-x}{y+x}$ find the value of y at $x=0.1$ using method. Given that $y(0)=1$

Given $\frac{dy}{dx} = \frac{y-x}{y+x}$

$$\boxed{f(x,y) = \frac{y-x}{y+x}}$$

Initial condition→ $x_0 = 0$

$y_0 = 1$

Apply the picard method

$$\boxed{y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\,dx} \quad n = 0, 1, 2, ---$$

First approximation →

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$= 1 + \int_{x_0}^{x} f(x,1)\,dx$$

$$= 1 + \int_0^x \frac{1-x}{1+x}\,dx$$

$$= 1 + \int_0^x \frac{1-x+1-1}{1+x}$$

$$= 1 + \int_0^x \frac{2-(1+x)\,dx}{1+x} = 1 + \int_0^x \frac{2}{1+x}\,dx - \int_0^x \frac{1+x}{1+x}\,dx$$

$$= 1 + \int_0^x \frac{2}{1+x}\,dx - \int_0^x dx$$

$$\boxed{\begin{aligned} y_1 &= 1+2\log(1+x) - x \\ y_1(0.1) &= 0.9828 \end{aligned}}$$

Second approximation →

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$= 1 + \int_0^x f(x, 1+2\log(1+x) - x)\,dx$$

$$= 1 + \int_0^x \frac{1+2\log(1+x)-x-x}{1+2\log(1+x)-x+x}$$

$$= 1 + \int_0^x \frac{1+2\log(1+x)-2x}{1+2\log(1+x)}\,dx$$

~~The value of above~~

The integration of above funct^n is difficult to integrate. Hence, the sol^n of the given D.E is

$$\boxed{y = 0.9828 \text{ at } x = 0.1} \quad Ans$$

Q→① obtain y when $x = 0.1$ & $x = 0.2$ Given that $\frac{dy}{dx} = x + y$; $y(0) = 1$ using picard formulla.

Q→② Solve $\frac{dy}{dx} = 1 + xy$ with $x_0 = 2$ & $y_0 = 0$ using picard method of successive approximation.

(1) $f(x,y) = x+y$

$$\left.\begin{array}{l} x_0 = 0 \\ y_0 = 1 \end{array}\right\} \text{initial condition}$$

$y = ?$ at $x = 0.1$ & 0.2

Apply Picard's formulla

$$\boxed{y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\,dx} \quad n = 0, 1, 2, \ldots$$

first approximation →

$$y_1 = y_0 + \int_0^x f(x, y_0)\,dx$$

$$= 1 + \int_{x_0}^{x} f(x,1)\,dx$$

$$= 1 + \int_{x_0}^{x} (x+1)\,dx$$

$$= 1 + \left[\frac{x^2}{2} + x\right]_0^x$$

$$y_1 = 1 + \frac{x^2}{2} + x$$

$$\boxed{y_1(0.1) = 1.105}$$

~~Second~~ ~~approxi~~

$$\boxed{y_1(0.2) = 1.220}$$

Second approximation →

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$= 1 + \frac{\cancel{2}x^2}{\cancel{2}} + x + \frac{x^3}{6}$$

$$= 1 + x^2 + x + x^3/6$$

$$\boxed{\begin{array}{l} y_2(0.1) = 1.110 \\ y_2(0.2) = 1.2431 \end{array}}$$

Third approximation →

$$y_3 = y_0 + \int_{x_0}^{x} (x, y_2)\,dx$$

$$= 1 + \int_{x_0}^{x} x + 1 + x^2 + x + \frac{x^3}{6}\,dx$$

$$= 1 + \int_{x_0}^{x} 1 + 2x + x^2 + \frac{x^3}{6}\,dx$$

$$= 1 + x + \frac{\cancel{2}x^2}{\cancel{2}} + \frac{x^3}{3} + \frac{x^4}{24}$$

$y(0.1) = 1.110$, $y(0.2) = 1.243$

fourth approximation→

$$y_4 = y_0 + \int_{x_0}^{x} (x, y_3)\, dx$$

$$= 1 + \int_{x_0}^{x} \left[x + 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right] dx$$

$$= 1 + \int_{x_0}^{x} \left[1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right] dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}$$

$y(0.1) = 1.110$, $y(0.2) = 1.243$ Ans

Since the value of last two approximation are same. Hence, the solⁿ is-

$$\boxed{\begin{array}{l} y = 1.110 \text{ at } x = 0.1 \\ y = 1.243 \text{ at } x = 0.2 \end{array}}$$

Ans

(2) Euler's method→

Suppose, $\frac{dy}{dx} = f(x, y)$; with initial condition x_0 &

$$y_1 = y_0 + h f(x_0, y_0)$$
$$y_2 = y_1 + h f(x_1, y_1)$$
$$y_3 = y_2 + h f(x_2, y_2)$$
$$\vdots$$
$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)} \quad n = 0, 1, 2, 3, \ldots$$

Q→ Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with $y=1$ for $x=0$. Find $y=1$ at

Euler's method.

Given $f(x, y) = \frac{y-x}{y+x}$

Initial condition→

$x_0 = 0$

$y_0 = 1$

$y = 1$ at $x = 0.1$

Apply Euler's method

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)} \quad n = 0, 1, 2, \ldots$$

$$y_1 = y_0 + h f(x_0, y_0)$$
$$= 1 + 0.1 f(0, 1)$$
$$= 1 + 0.1 \left(\frac{1-0}{1+0}\right)$$
$$= 1 + 0.1 \times 1$$

$x_1 = x_0 + h$

$x_2 = x_1 + h$

$y_1 = 1.1$ at $x_1 = 0.1$ $\because \left[(x_0 + h) = 0 + 0.1\right]$ Ans

Q→ Solve the eqn $\frac{dy}{dx} = 1 - y$ with the initial condition

Euler method.

Given $f(x, y) = 1 - y$

Initial condition→

$x_0 = 0$

$y_0 = 0$

$y = 1$ at $x = 0.1, 0.2, 0.3$

$y(0) =$

$y(x) =$

$y(0) =$

$x_0 =$

$y_0 =$

Apply Euler's formula

$$\boxed{y_{n+1} = y_n + hf(x_n, y_n)} \quad n = 0, 1, 2, \dots n$$

$$y_1 = y_0 + hf(x_0, y_0)$$
$$= 0 + 0.1 f(0, 0)$$
$$= 0 + 0.1$$
$$= 0.1$$

$y_1 = 0.1$ at $x_1 = 0.1$ Ans

$$y_2 = y_1 + hf(x, y_1)$$
$$= 0.1 + 0.1 f(0.1, 0.1)$$
$$= 0.1 + 0.1(1 - 0.1)$$
$$= 0.1 + 0.1(0.9) = 0.19$$

$y_2 = 0.19$ at $x_2 = 0.2$ Ans

$$y_3 = y_2 + hf(x_2, y_2)$$
$$= 0.19 + 0.1 f(0.2, 0.19)$$
$$= 0.19 + 0.1(1 - 0.19)$$
$$= 0.19 + 0.1(0.81)$$
$$= 0.271$$

$y_3 = 0.271$ at $x_3 = 0.3$ Ans

Hence, the solⁿ is

x	y
0	0
0.1	0.1
0.2	0.19
0.3	0.271

Ans

R.K. Method (R.K. 4th order) (Runge-Kutta Method)

Suppose, $\frac{dy}{dx} = f(x, y)$ with initial condition x_0 & y_0

$$k_1 = h f(x_0, y_0)$$

$$k_2 = h f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h f(x_0 + h,\ y_0 + k_3)$$

Finally, we compute k

$$\boxed{k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$y_1 = y_0 + k \quad \text{at } x_1 = x_0 + h$$

Q→ Using R-K method to solve the following differential $\frac{dy}{dx} = x + y$ with $y(0) = 1$ & find $y = ?$ at $x = 0.2$.

Given: $f(x, y) = x + y$

initial condition: $x_0 = 0$

$y_0 = 1$

$h = 0.2$

Apply the R.K. method:

$$k_1 = h f(x_0, y_0)$$

$$= 0.2 f(0, 1)$$

$$= 0.2 (0 + 1)$$

$$= 0.2$$

$$\boxed{k_1 = 0.2}$$

$$k_2 = h f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$$

$$= 0.2 f\left(0 + \frac{0.2}{2},\ 1 + \frac{0.2}{2}\right)$$

$$= 0.2 (0.1, 1.1)$$

$$= 0.2 (0.1 + 1.1) = 0.2 \times 1.2 = 0.2400$$

$$\boxed{k_2 = 0.2400}$$

$$k_3 = h f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right)$$

$$= 0.2 f\left(0 + \frac{0.2}{2},\ 1 + \frac{0.24}{2}\right)$$

$$= 0.2 (0.1 + 1.1200)$$

$$= 0.2440$$

$$\boxed{k_3 = 0.2440}$$

$k_4 = hf(x_0+h, y_0+k_3)$

$= 0.2f(0+0.2, 1+0.2440)$

$= 0.2(0.2+1.2440)$

$$\boxed{k_4 = 0.2888}$$

finally, we compute k

$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}(0.2 + 2\times0.2400 + 2\times0.2440 + 0.2888)$

$$\boxed{k = 0.2428}$$

$y_1 = y_0 + k = 1 + 0.2428$

$= 1.2428$

at $x_1 = x_0 + h = 0 + 0.2$

$= 0.2$

$y_1 = 1.2428$ at $x_1 = 0.2$ Ans

Q→ using R.K. method of 4th order solve the following differential eqn

$$\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}, \quad y(0) = 1 \text{ at } x = 0.2, 0.4.$$

Given $f(x,y) = \frac{y^2 - x^2}{y^2 + x^2}$

initial condition: $x_0 = 0$

$y_0 = 1$

$h = 0.2$

Apply the R.K. method:

$k_1 = hf(x_0, y_0)$

$= 0.2 f(0,1)$

$= 0.2\left(\frac{1-0}{1+0}\right)$

$$\boxed{k_1 = 0.2}$$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= 0.2f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right)$

$= 0.2(0.1, 1.1)$

$$\boxed{k_2 = 0.1967}$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right)$$
$$= 0.2 f\left(0 + \frac{0.2}{2},\ 1 + \frac{0.1967}{2}\right)$$
$$= 0.2\ (0.1,\ 1.0984)$$
$$\boxed{k_3 = 0.1967}$$

$$k_4 = hf\ (x_0 + h,\ y_0 + k_3)$$
$$= 0.2\ f(0 + 0.2,\ 1 + 0.1967)$$
$$= 0.2\ (0.2,\ 1.1967)$$
$$\boxed{k_4 = 0.1891}$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}(0.2 + 2\times 0.1967 + 2\times 0.1967 + 0.1891)$$
$$\boxed{k = 0.1960}$$

$$y_1 = y_0 + k = 1 + 0.1960 = 1.1960$$
$$x_1 = x_0 + h = 0 + 0.2 = 0.2$$
$$y_1 = 1.1960 \text{ at } x_1 = 0.2 \ \text{Ans}$$

Again, apply R.K. method: initial condition:

$$y_1 = 1.1960$$
$$x_1 = 0.2$$
$$h = 0.2$$
$$k_1 = hf\ (x_1,\ y_1)$$
$$= 0.2 f(0.2,\ 1.1960)$$
$$= 0.2\left[\frac{1.4304 - 0.0400}{1.4304 + 0.0400}\right]$$
$$= 0.2\left[\frac{1.3904}{1.4704}\right] = 0.1891$$
$$\boxed{k_1 = 0.1891}$$

$$k_2 = 0.2 f\ (x_1,\ y_1)$$
$$= 0.2 f\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_1}{2}\right)$$
$$= 0.2\left(0.2 + \frac{0.2}{2},\ 1.1960 + \frac{0.1891}{2}\right)$$
$$= 0.2\ (0.3,\ 1.2906)$$
$$\boxed{k_2 = 0.1795}$$

$k_3 = 0.2 f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$

$= 0.2 f\left(0.3, 1.1960 + \frac{0.1795}{2}\right)$

$= 0.2 \,(0.3, 1.2858)$

$\boxed{k_3 = 0.1793}$

$k_4 = 0.2 f(x_1 + h, y_1 + k_3)$

$= 0.2 f(0.2 + 0.2, 1.1960 + 0.1793)$

$= 0.2\,(0.4, 1.3753)$

$\boxed{k_4 = 0.1688}$

$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\boxed{k = 0.1793}$

$y_2 = y_1 + k$ at $x_2 = x_1 + h$

$= 1.1960 + 0.1793$ at $x_2 = 0.2 + 0.2$

$y_2 = 1.3753$ at $x_2 = 0.4$

Q → Use classical R.K. method by 4th order to find the numerical sol^n at $x = 1.4$ for $\frac{dy}{dx} = y^2 + x^2$, $y(1) = 0$. Assume step size $h = 0.2$.

Taylor's method →

$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \cdots$ at $x_1 = x_0 + h$

$y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \cdots$ at $x_2 = x_1 + h$

$\vdots$

$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!} y_n'' + \cdots$ at $x_n = x_{n-1} + h \quad (n = 0, 1, 2, \cdots)$

Q → Use Taylor series method to solve numerically $\frac{dy}{dx} = x + y$ with $y(1) = 0$ at $x = 1.2$ using $h = 0.1$.

Given $\frac{dy}{dx} = x + y$

$$\boxed{y' = x + y}$$

$x_0 = 1,\ y_0 = 0,\ h = 0.1$

$y = ?$ at $x = 1.2$

Apply the Taylor series

$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!} y_n'' + \cdots$ at $x_n = x_{n-1} + h \quad (n = 0, 1, 2, \cdots)$

$y_1 = y_0 + hy_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{3!} y_0''' + \cdots\ x_1 = x_0 + h$

$y' = x + y \Rightarrow y_0' = 1 + 0 = 1$

$y'' = 1 + y' \Rightarrow y_0'' = 1 + y_0' = 1 + 1 = 2$

$y''' = 0 + y'' \Rightarrow y_0''' = y_0'' = 2$

$y_1 = y_0 + hy_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{6} y_0''' + \cdots$

$= 0 + 0.1 \times 1 + \frac{(0.1)^2}{2} \times 2 + \frac{(0.1)^3}{6} \times 2$

$y_1 = 0.1103$ at $x_1 = x_0 + h = 1 + .1 = 1.1$

$y_2 = y_1 + hy_1' + \frac{h^2}{2} y_1'' + \frac{h^3}{6} y_1''' + \cdots$ at $x_2 = x_1 + h$

$y' = x_1 + y_1 \Rightarrow y_1' = 1.1 + 0.1103 = 1.2103$

$\Rightarrow y_1'' = 1 + y_1' = 1 + 1.2103 = 2.2103$

$\Rightarrow y_1''' = 0 + y_1'' = 0 + 2.2103 = 2.2103$

$y_2 = 0.1103 + 0.1 \times 1.2103 + \frac{(0.1)^2}{2} \times 2.2103 + \frac{(0.1)^3}{6} \times 2.2103$

$$\boxed{y_2 = 0.2427 \text{ at } x_2 = 1.2}$$

Q → from the Taylor series find $y(0\cdot1)$ correct to 4 decimal places. If $y(x)$ satisfy $\frac{dy}{dx} = x - y^2$ & $y(0) = 1$ Also find $y(0\cdot2)$.

Given → $y' = x - y^2$

$x_0 = 0,\ y_0 = 1,\ h = 0\cdot1$

$y' = x - y^2 \Rightarrow y_0' = 0 - 1 = -1$

$y'' = 1 - 2y \cdot y' \Rightarrow y_0'' = 1 - 2 \times 1\ (-1) = 3$

$y''' = -2(y \cdot y'' + y'y') \Rightarrow y_0''' = -2\ (1 \times 3 + (-1)(-1))$

$= -8$

$y_1 = y_0 + hy_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{6} y_0'''$

$= 1 + 0\cdot1 \times (-1) + \frac{(0\cdot1)^2}{2} \times 3 + \frac{(0\cdot1)^3}{6} \times (-8)$

$y_1 = 0\cdot9137$ at $x_1 = x_0 + h = 0 + 0\cdot1 = 0\cdot1$

$y_2 = y_1 + hy_1' + \frac{h^2}{2} y_1'' + \frac{h^3}{6} y_1'''$

$y' = x - y^2 \Rightarrow y_1' = 0\cdot1 - (0\cdot9137)^2 = -0\cdot7348$

$\Rightarrow y_1'' = 1 - 2y \cdot y' \Rightarrow 1 - 2 \times 0\cdot9137 \times (-0\cdot7348)$

$= 2\cdot3428$

$\Rightarrow y_1''' = -2(y \cdot y'' + y'y') \Rightarrow -2(0\cdot9137 \times 2\cdot3428) + (-0\cdot7348)(-0\cdot7348)$

$y_1''' = -5\cdot3611$

$y_2 = 0\cdot9137 + 0\cdot1 \times (-0\cdot7348) + \frac{(0\cdot1)^2}{2} \times 2\cdot3428 + \frac{(0\cdot1)^3}{6} \times (-5\cdot3611)$

$y_2 = 0\cdot8510$ at $x_2 = x_1 + h = 0\cdot1 + 0\cdot1$

$\boxed{y_2 = 0\cdot8510 \text{ at } x_2 = 0\cdot2}$ A

Predictor - Corrector method:

Milne's Predictor - Corrector method:

$$y^p_{n+1} = y_{n-3} + \frac{4h}{3}\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$$

$$y^c_{n+1} = y_{n-1} + \frac{h}{3}\left(y'_{n-1} + 4y'_n + y^{p'}_{n+1}\right)$$

where $n = 3, 4, 5, 6, \ldots$

$$y^p_4 = y_0 + \frac{4h}{3}\left(2y'_1 - y'_2 + 2y'_3\right)$$

$$y^c_4 = y_2 + \frac{h}{3}\left(y'_2 + 4y'_3 + y^{p'}_4\right)$$

Q: Use Milne's predictor - corrector method solve

D.E. $\frac{dy}{dx} = 1 + y^2$ $\qquad y' = 1 + y^2$

with $y(0) = 0$
$y(0.2) = 0.2007$
$y(0.4) = 0.4228$
$y(0.6) = 0.6841$
obtain $y(0.8)$.

Given $\frac{dy}{dx} = 1 + y^2$

$y' = 1 + y^2$

$x_0 = 0$
$y_0 = 0$

$x_1 = 0.2$
$y_1 = 0.2007 \Rightarrow \boxed{y'_1 = 1 + y_1^2 = 1.0403}$

$x_2 = 0.4$
$y_2 = 0.4228 \Rightarrow \boxed{y'_2 = 1 + y_2^2 = 1.1788}$

$x_3 = 0.6$
$y_3 = 0.6841 \Rightarrow \boxed{y'_3 = 1.4680}$

Step(1): Now we find the predictor value

$$y^p_4 = y_0 + \frac{4h}{3}\left(2y'_1 - y'_2 + 2y'_3\right)$$

$$= 0 + \frac{4 \times 0.2}{3}\left(2 \times 1.0403 - 1.1788 + 2 \times 1.4680\right)$$

$$\Rightarrow \boxed{y^p_4 = 1.0234}$$

Step(2) → Now, we find the corrector value

$$y_4^c = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4^{p'}\right)$$

$$\boxed{y_4^{p'} = 1 + y_4^{p^2} = 2.0473}$$

$$y_4^c = 0.4228 + \frac{0.2}{3}(1.1788 + 4 \times 1.4680 + 2.0475$$

$$\boxed{y_4^c = 1.0293}$$